Wave focusing and dispersion in unidirectional and directionally spreading sea in deep water

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Overview

Background

- Wave focusing and dispersion
- NLS equations and conservation laws

Derivation of analytical models

- Unidirectional cases
- Directional cases

Model validation and comparisons

- Bandwidth change
- Wave shape modification

Further discussions

- Asymmetric features

Wave focusing and dispersion

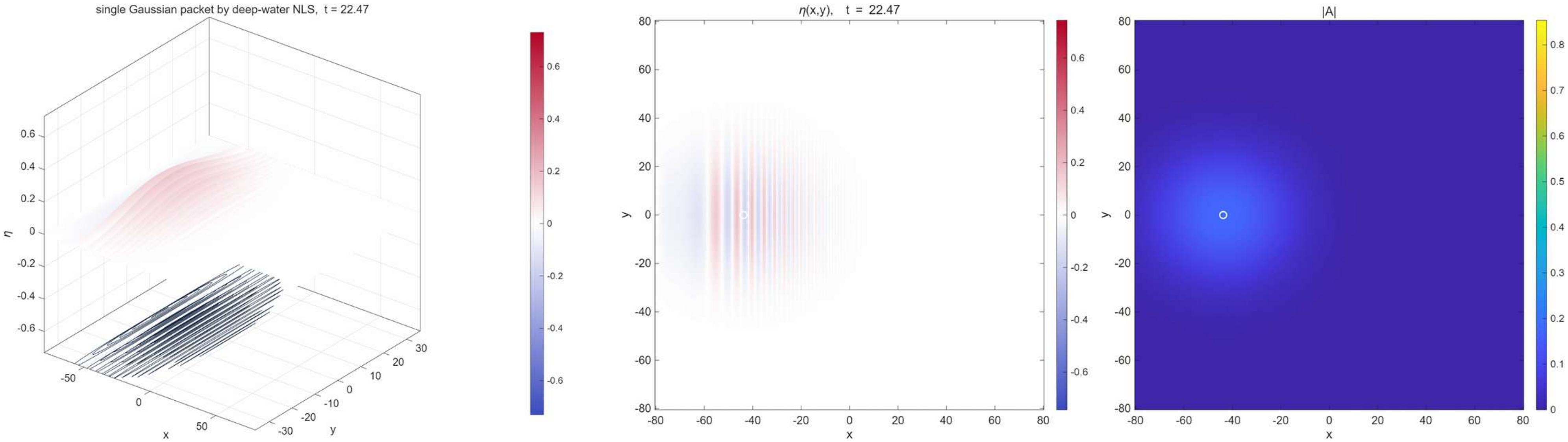

Fig 1. Directional wave focusing and dispersion: (a) 3D surface elevation; (b) surface elevation; (c) amplitude of wave packet.

At focusing, cubic wave nonlinearity can:

- Increase the amplitude and narrow the packet (in physical domain);
- Increase the bandwidth (in Fourier space).

After focusing events, dispersion effects can:

- broaden wave packet and reduce amplitude;
- Decrease the bandwidth.

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NLS equations with conservation laws

To better understand the combined effects of nonlinearity and dispersion, and to quantify the evolution of bandwidth and amplitude change.

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} + |U|^2 U.$$

as it holds several conservation laws, e.g. energy and Hamiltonian conservation

$$I2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |U|^2 \,\mathrm{d}X\,\mathrm{d}Y$$

$$I4 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|U_X|^2 - |U_Y|^2 - \frac{1}{2}|U|^4\right)\mathrm{d}X\,\mathrm{d}Y$$

By assuming the evolution of wave packet follows the linearized solution of the NLS equation (Kinsman, 1965),

amplitude

bandwidth

$$U(X,Y,T) = \frac{A}{\sqrt{1-2\mathrm{i}S_X^2 T}\sqrt{1-2\mathrm{i}S_Y^2 T}}\exp\left(-\frac{1}{2}\frac{S_X^2 X^2}{1-2\mathrm{i}S_X^2 T}\right)\exp\left(-\frac{1}{2}\frac{S_Y^2 Y^2}{1-2\mathrm{i}S_Y^2 T}\right)$$

While allowing key parameters, i.e. A, SX, SY, to change with time, A(T), SX(T), SY(T) .

Analytical model for unidirectional cases (Adcock and Taylor, 2009)

$$\frac{A_\infty^2}{S_\infty} = \frac{A_\mathrm{f}^2}{S_\mathrm{f}}$$

$$A_\infty^2 S_\infty = A_\mathrm{f}^2 S_\mathrm{f} - \frac{1}{\sqrt{2}}\frac{A_\mathrm{f}^4}{S_\mathrm{f}}$$

The realistic sea is usually broader-banded with stronger nonlinearity (Van den Bremer and Taylor 2016)

- Higher-order dispersion (bandwidth)
- Bound harmonics, e.g. return flow

Analytical model for directional cases (Adcock et al., 2012)

$$\frac{A_\mathrm{f}^2}{S_{X\mathrm{f}} S_{Y\mathrm{f}}} = \frac{A_\infty^2}{S_{X\infty} S_{Y\infty}}$$

$$\frac{\pi A_\mathrm{f}^2 (S_{X\mathrm{f}}^2 - S_{Y\mathrm{f}}^2)}{2 S_{X\mathrm{f}} S_{Y\mathrm{f}}} - \frac{\pi A_\mathrm{f}^4}{4 S_{X\mathrm{f}} S_{Y\mathrm{f}}} = \frac{\pi A_\infty^2 (S_{X\infty}^2 - S_{Y\infty}^2)}{2 S_{X\infty} S_{Y\infty}}$$

Modified NLS equations for broader-banded sea-states

Trulsen et al. (2000) derived a modified nonlinear Schrödinger (MNLS) equation to incorporate an exact linear dispersion and return-flow

$$\frac{\partial u}{\partial t} + \underbrace{\mathcal{L}(\partial_x, \partial_y)u}_{\text{Exact dispersion}} + \mathrm{i}\frac{\omega_0 k_0^2}{2}|u|^2 u + \frac{3\omega_0 k_0}{2}|u|^2\frac{\partial u}{\partial x} + \frac{\omega_0 k_0}{4}|u|^2\frac{\partial u^*}{\partial x} + \underbrace{\mathrm{i}k_0 u \left.\frac{\partial \bar{\phi}}{\partial x}\right|_{z=0}}_{\text{Return flow}} = 0$$

$$\mathcal{L}(\partial_x, \partial_y) = \mathrm{i}\sqrt{g}\left[\left((k_0 - \mathrm{i}\partial_x)^2 - \partial_y^2\right)^{1/4} - \sqrt{k_0}\right] \qquad \left.\frac{\partial \bar{\phi}}{\partial x}\right|_{z=0} = \frac{1}{2\pi}\int \frac{-\omega_0 k_x^2}{2|\mathbf{k}|}\mathcal{F}_{\mathbf{x}}(|u|^2)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\,d\mathbf{k}$$

Li (2021) extended the model of Trulsen et al. (2000) to allow the nonlinear terms not limited to the narrow-banded assumption.

$$\partial_t A + \mathrm{i}\omega_0 \mathcal{L} A + \mathcal{N}(\boldsymbol{x}, z, t) = 0$$

$$\mathcal{N}_{all} = \frac{1}{g}(\mathrm{i}\omega_0 - \partial_t)[V^{(20)}\cdot\bar{V}] - \frac{1}{8g}[2k_0\bar{w} + \bar{V}\cdot\nabla_3 + 4k_0 A\partial_t](\bar{V}\cdot\bar{V}^*)$$

$$\mathcal{N}_{dir} = \frac{1}{g}(\mathrm{i}\omega_0 - \partial_t)(\bar{V}^{(22)}\cdot\bar{V}^*) - \frac{1}{16g}[(2k_0^{(3d)} + \nabla_3)(|\bar{V}|^2)]\cdot\bar{V}$$

Although these models can help better understand wave hydrodynamics in broad-banded sea-states, they are non-Hamiltonian.

Gramstad and Trulsen (2011) applied canonical transformation to convert the Trulsen et al.'s (2000) to become Hamiltonian:

$$I_4 = \int\left[v^*\mathcal{L}(\partial_x, \partial_y)\,v + \frac{\mathrm{i}\omega_0 k_0^2}{4}|v|^4 + \frac{3\omega_0 k_0}{8}|v|^2\left(v\frac{\partial v^*}{\partial x} - v^*\frac{\partial v}{\partial x}\right) + \frac{\mathrm{i}\omega_0}{4}|v|^2\left.\frac{\partial\bar{\phi}}{\partial x}\right|_{z=0}\right]\mathrm{d}x$$

$$u = \left[k_0^{-1}\sqrt{(k_0 - \mathrm{i}\partial_x)^2 - \partial_y^2}\right]^{1/4} v = v + \frac{\mathrm{i}}{4k_0}\frac{\partial v}{\partial x} + \ldots$$

To extend the models of Adcock and Taylor (2009) and Adcock et al. (2012) to quantify the influence of higher-order dispersion and return flow on key parameters in wave focusing and dispersion.

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The present model-unidirectional waves

The dispersion effects are expanded to fourth-order accuracy.

$$\widetilde{\mathcal{L}} \approx -\mathrm{i}\sqrt{2}\frac{\partial}{\partial X} + \frac{\partial^2}{\partial X^2} + \mathrm{i}\sqrt{2}\frac{\partial^3}{\partial X^3} - \frac{5}{2}\frac{\partial^4}{\partial X^4}$$

Then, substitute the following solution

$$U(X,T) = \frac{A}{\sqrt{1-2\mathrm{i}S_X^2 T}} \exp\left(-\frac{1}{2}\frac{S_X^2 X^2}{1-2\mathrm{i}S_X^2 T}\right)$$

to the conservation laws

$$I_2 = \int |U|^2 \,\mathrm{d}\mathbf{X}$$

$$I_4 = \int \left\{ \left|\frac{\partial U}{\partial X}\right|^2 + \underbrace{\boxed{\frac{5}{2}\left|\frac{\partial^2 U}{\partial X^2}\right|^2}}_{\text{Fourth-order bandwidth}} - \frac{1}{2}|U|^4 + \underbrace{\boxed{\sqrt{2}|U|^2\Phi_X(|U|^2)}}_{\text{Return flow}} \right\} \mathrm{d}\mathbf{X}$$

We have

$$I_2 = \frac{\sqrt{\pi}A^2}{S_X}$$

$$I_4 = \frac{\sqrt{\pi}}{2}A^2 S_X + \underbrace{\frac{15\sqrt{\pi}}{8}A^2 S_X^3}_{\text{fourth-order bandwidth}} \overbrace{- \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{A^4}{S_X\sqrt{1+4S_X^4T^2}}}^{\text{wave nonlinearity}} + \underbrace{\frac{\sqrt{2}A^4}{1+4S_X^4T^2}}_{\text{return flow}}$$

The present analytical model for unidirectional cases

$$\begin{cases} \dfrac{A_f^2}{S_{Xf}} = \dfrac{A_\infty^2}{S_{X\infty}}, \\ A_f^2 S_{Xf} + \dfrac{15}{4}A_f^2 S_{Xf}^3 - \dfrac{1}{\sqrt{2}}\dfrac{A_f^4}{S_{Xf}} + 2\sqrt{\dfrac{2}{\pi}}A_f^4 = A_\infty^2 S_{X\infty} + \dfrac{15}{4}A_\infty^2 S_{X\infty}^3 \end{cases}$$

which relate four parameters, A_{Xf} and S_{Xf} at focusing and $A_{X\infty}$ and $S_{X\infty}$ at fully dispersive stage.

Closed-form solution

$$\left(\frac{S_{X\infty}}{S_{Xf}}\right)^2 = \frac{2}{15S_{Xf}^2}\left[-1 + \frac{1}{2}\sqrt{\left(2+15S_{Xf}^2\right)^2 + 30\sqrt{2}S_{Xf}^2\left(-1+\frac{4S_{Xf}}{\sqrt{\pi}}\right)\left(\frac{A_f}{S_{Xf}}\right)^2}\right]$$

Here, our analysis considers bandwidth up-to fourth-order accuracy, while it can be expanded to arbitrary-order.

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The present model-directional waves

The dispersion effects are expanded to fourth-order accuracy.

$$\widetilde{\mathcal{L}} \approx -\mathrm{i}\sqrt{2}\frac{\partial}{\partial X}+\left(\frac{\partial^2}{\partial X^2}-\frac{\partial^2}{\partial Y^2}\right)+\mathrm{i}\sqrt{2}\frac{\partial}{\partial X}\left(\frac{\partial^2}{\partial X^2}-3\frac{\partial^2}{\partial Y^2}\right)+\frac{1}{2}\left(-5\frac{\partial^4}{\partial X^4}+30\frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2}-3\frac{\partial^4}{\partial Y^4}\right)$$

Then, substitute the following solution

$$U(X,Y,T)=\frac{A}{\sqrt{1-2\mathrm{i}S_X^2T}\sqrt{1-2\mathrm{i}S_Y^2T}}\exp\left(-\frac{1}{2}\frac{S_X^2X^2}{1-2\mathrm{i}S_X^2T}\right)\exp\left(-\frac{1}{2}\frac{S_Y^2Y^2}{1-2\mathrm{i}S_Y^2T}\right)$$

to the conservation laws. The return flow cannot be solved explicitly, requiring an approximation for the kernel

$$\Phi_X(|U|^2)=\frac{A^2}{2S_XS_Y}\mathcal{F}_{\mathbf{K_X}}^{-1}\left\{\boxed{\frac{K_X^2}{\sqrt{K_X^2+K_Y^2/2}}}\exp\left[-K_X^2\left(\frac{1}{4S_X^2}+S_X^2T^2\right)-K_Y^2\left(\frac{1}{4S_Y^2}+S_Y^2T^2\right)\right]\right\}$$

dominant direction

$$\frac{K_X^2}{\sqrt{K_X^2+K_Y^2/2}}\approx\frac{|K_X|}{\sqrt{1+\lambda^2/2}},\quad \text{and} \quad \lambda=\sqrt{\left(\frac{1}{4S_Y^2}+S_Y^2T^2\right)\left(\frac{1}{4S_X^2}+S_X^2T^2\right)^{-1}}$$

Thus, an approximation solution of the return flow is

$$\Phi_X(|U|^2)\approx\frac{2A^2S_X}{\sqrt{1+\lambda^2/2}\sqrt{\pi}}\frac{\exp\left[-\left(1+4S_Y^4T^2\right)^{-1}S_Y^2Y^2\right]}{\left(1+4S_X^4T^2\right)^{3/2}\sqrt{1+4S_Y^4T^2}}\left[\sqrt{1+4S_X^4T^2}-2S_XX\,\mathrm{DF}\left(\frac{S_XX}{\sqrt{1+4S_X^4T^2}}\right)\right]$$

$$\mathrm{DF}(x)=\int_0^x\exp(\varsigma^2-x^2)\,\mathrm{d}\varsigma$$

The present analytical model for directional cases

$$\begin{cases}\dfrac{\pi A^2}{S_{Xf}S_{Yf}}=\dfrac{\pi A^2}{S_{X\infty}S_{Y\infty}},\\[2ex] \dfrac{\pi A^2(S_{Xf}^2-S_{Yf}^2)}{2S_{Xf}S_{Yf}}+\dfrac{15\pi A^2S_{Xf}^3}{8S_{Yf}}+\dfrac{9\pi A^2S_{Yf}^3}{8S_{Xf}}-\dfrac{\pi A^4}{4S_{Xf}S_{Yf}}+\dfrac{\sqrt{\pi}}{\sqrt{1+\lambda^2/2}}\dfrac{A^4}{S_{Yf}}\\ \qquad=\dfrac{\pi A^2(S_{X\infty}^2-S_{Y\infty}^2)}{2S_{X\infty}S_{Y\infty}}+\dfrac{15\pi A^2S_{X\infty}^3}{8S_{Y\infty}}+\dfrac{9\pi A^2S_{Y\infty}^3}{8S_{X\infty}}.\end{cases}$$

The reason why we discuss unidirectional and directional cases separately:

1. Three parameters (A, SX, SY) v.s. two conservation laws (non-trivial). The momentum conservation is trivial to a symmetric Gaussian solution.

2. One parameter needs to be frozen, here, we choose A to be a constant. Compared to the variation of bandwidth, the amplitude is relatively smaller, <6%, (Gibbs and Taylor, 2005). (We will validate this assumption later).

3. The model for directional wave will not reduce to that for unidirectional wave.

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Unidirectional waves-amplitude ratio

The carrier-wave period and wavenumber are set to T_p= 12s and k_p = 0.0279/m, respectively, representative of a North Sea storm. Let waves start from the focusing stage (t=0) with a Gaussian input and become fully dispersive at t= ∞.

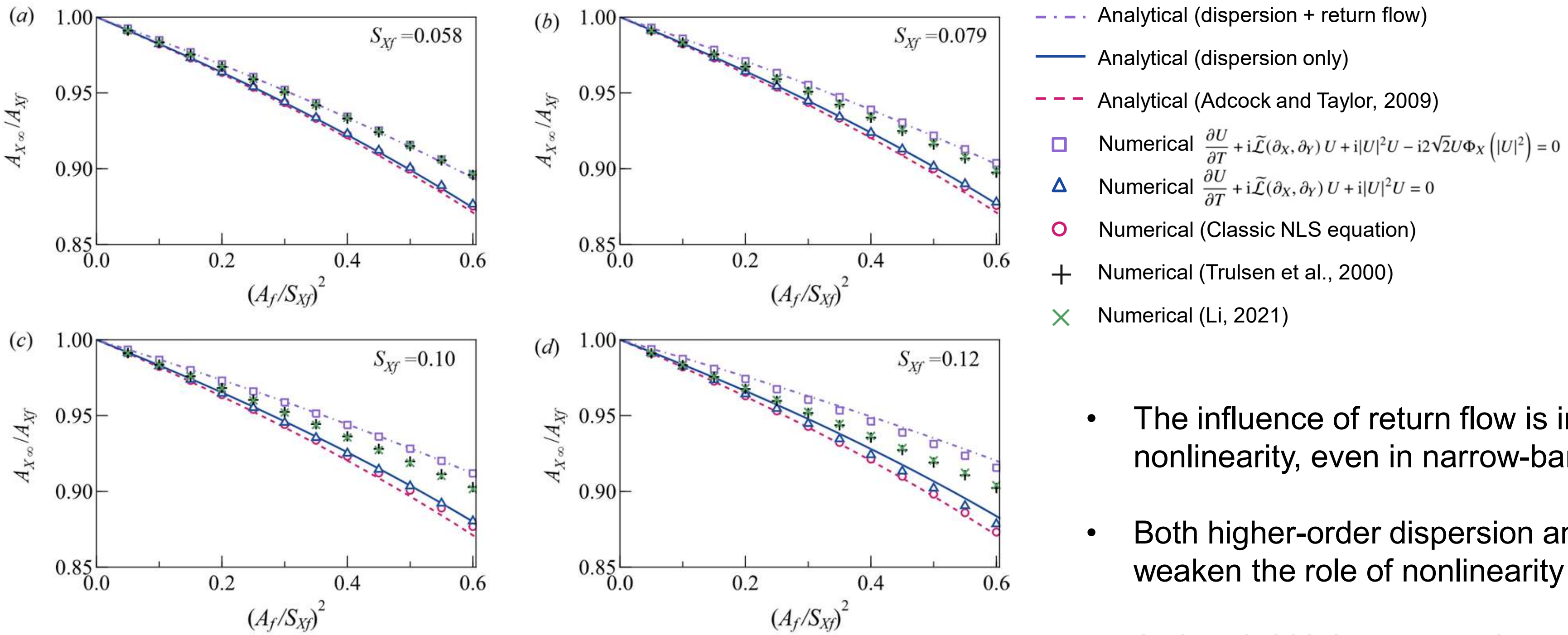

Fig 2. Comparison of the amplitude ratio, $A_{X\infty} / A_{Xf}$, between the fully dispersed and focusing states for increasing nonlinearity and different initial bandwidths: (a) S_{Xf}=0.058, (b) S_{Xf}=0.079, (c) S_{Xf}=0.10, and (d) S_{Xf}=0.12.

- The influence of return flow is important with increasing nonlinearity, even in narrow-banded sea.
- Both higher-order dispersion and return flow tend to weaken the role of nonlinearity (cubic nonlinear term).
- As bandwidth increases, the results overestimate the bandwidth ratio, which is induced by the asymmetry of the wave groups.

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Unidirectional waves-application for predicting focusing

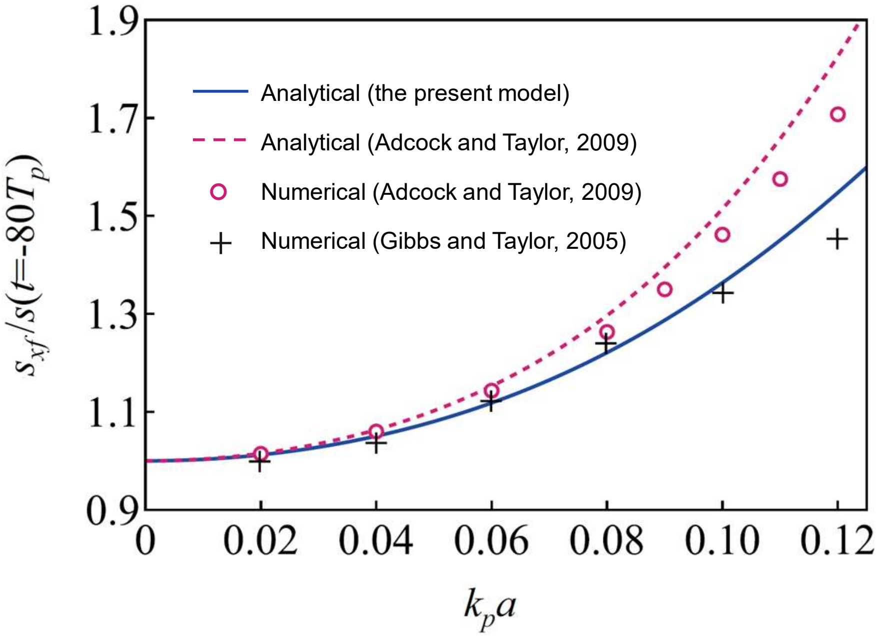


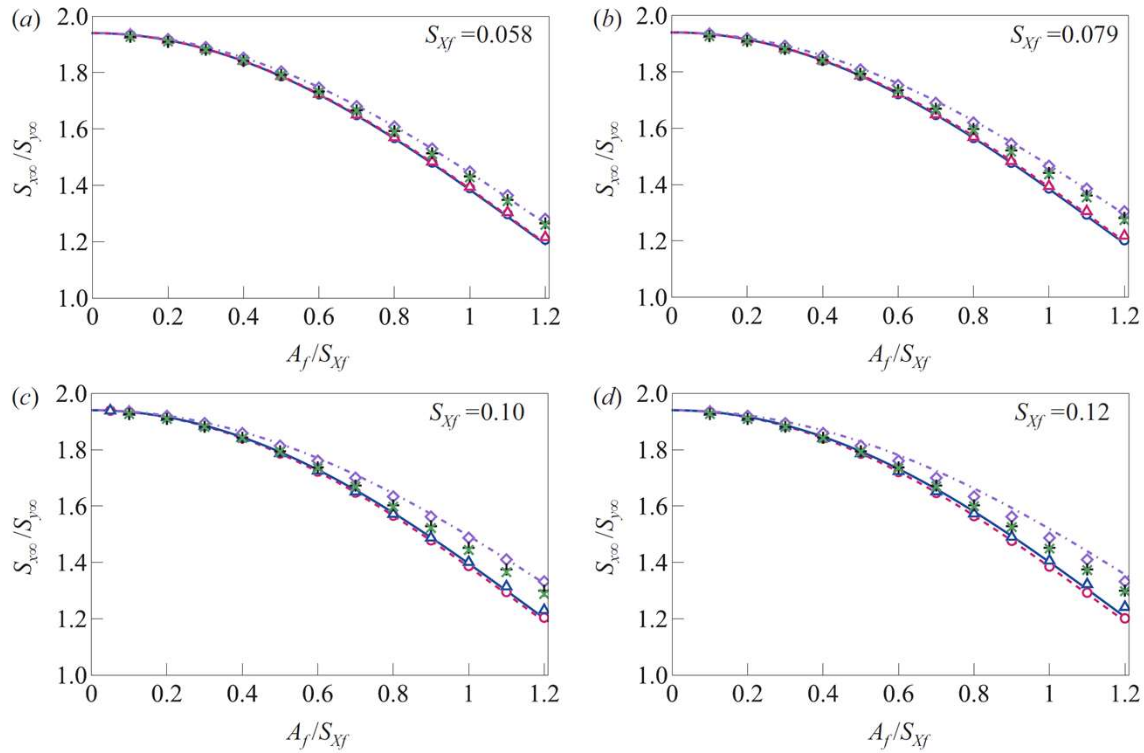
Fig 3. Comparison of the amplitude ratio, $A_{X\infty}$ / A_{Xf} , between the fully dispersed and focusing states for increasing nonlinearity and different initial bandwidths:(a) SXf =0.058, (b) SXf =0.079, (c) SXf =0.10, and (d) SXf =0.12.

We study the case considered by Gibbs and Taylor (2005), where the evolution of an initial Gaussian wave group at $T = -80T$p before focusing at $T = 0$ was investigated.

$$\begin{cases} \dfrac{A_f^2}{S_{Xf}} = \dfrac{A^2(T)}{S_X(T)}, \\ A_f^2 S_{Xf} + \dfrac{15}{4} A_f^2 S_{Xf}^3 - \dfrac{1}{\sqrt{2}} \dfrac{A_f^4}{S_{Xf}} + 2\sqrt{\dfrac{2}{\pi}} A_f^4 \\ \quad = A^2(T) S_X(T) + \dfrac{15}{4} A^2(T) S_X^3(T) - \dfrac{1}{\sqrt{2}} \dfrac{A^4(T)}{S_X(T)\sqrt{1 + 4S_X^4(T)T^2}} + 2\sqrt{\dfrac{2}{\pi}} \dfrac{A^4(T)}{1 + 4S_X^4(T)T^2} \end{cases}$$

- The difference between the results of Adcock and Taylor (2009) and Gibbs and Taylor (2005) is possibly attributed to the return flow effects, which can be captured by the present model.

Directional waves-bandwidth ratio



- - · - Analytical (dispersion + return flow)
- — Analytical (dispersion only)
- - - - Analytical (Adcock and Taylor, 2009)
- □ Numerical $\frac{\partial U}{\partial T} + \mathrm{i}\tilde{\mathcal{L}}(\partial_X, \partial_Y)\,U + \mathrm{i}|U|^2U - \mathrm{i}2\sqrt{2}U\Phi_X\left(|U|^2\right) = 0$
- △ Numerical $\frac{\partial U}{\partial T} + \mathrm{i}\tilde{\mathcal{L}}(\partial_X, \partial_Y)\,U + \mathrm{i}|U|^2U = 0$
- ○ Numerical (Cubic NLS equation)
- + Numerical (Trulsen et al., 2000)
- × Numerical (Li, 2021)

Fig 4. Comparison of the bandwidth ratio in the fully dispersed state, $S_{x\infty}$ / $S_{y\infty}$, between the analytical and numerical solutions with increasing nonlinearity for S_{xf} / S_{yf} = 1.94 and different initial bandwidths S_{xf}: (a) S_{Xf} =0.058, (b) S_{Xf} =0.079, (c) S_{Xf} =0.10, and (d) S_{Xf} =0.12.

- The relative error in all numerical solutions, with $|A_{\mathrm{nonlinear}}-A_{\mathrm{linear}}|/A_{\mathrm{linear}}$ <3%, indicates that the assumption of A being frozen is reasonable.
- Both higher-order dispersion and return flow can weaken the influence of nonlinearity by increasing the bandwidth ratio of $S_{x\infty}$ / $S_{y\infty}$.

Asymmetric features for broader bandwidth conditions

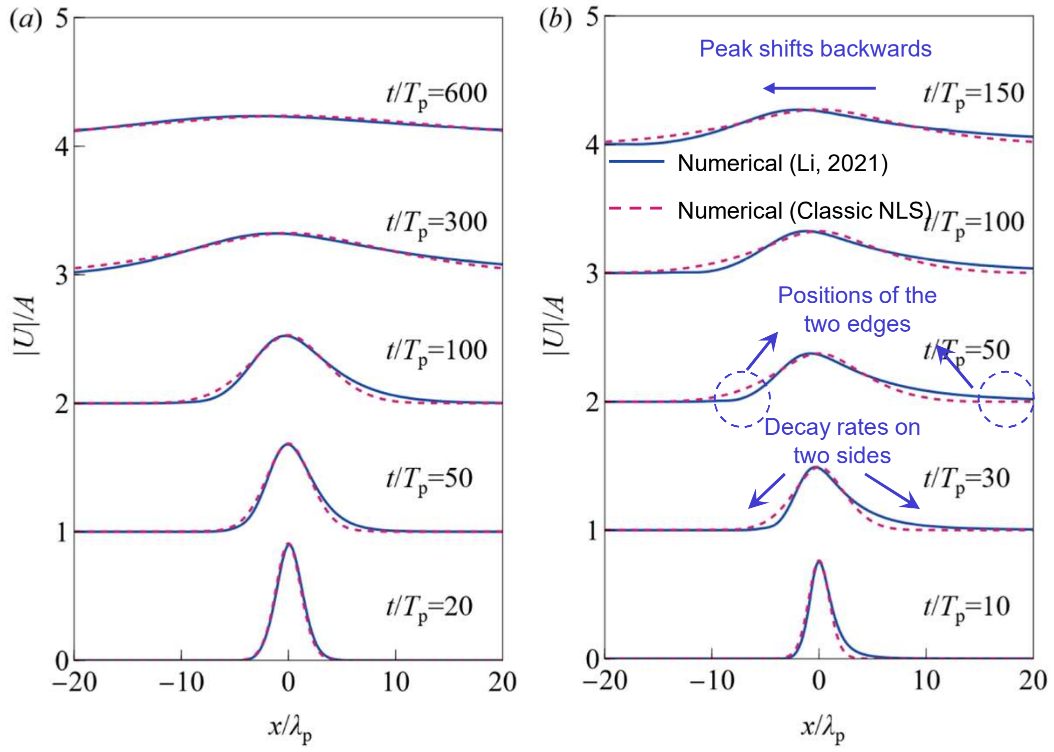


Fig 5. Comparison of the temporal evolution of the wave packet predicted by the model of Li (2021) (blue solid line) and classic NLS equation (red dashed line), for A_f/S_{xf}=0.77. (a) S_{Xf}=0.058, (b) S_{Xf}=0.079.

For broader initial bandwidth, asymmetric features can be observed in numerical simulations

- Peak shifts backwards
- Different positions of the two edges of the packet
- Different decay rates on two sides of the peak

In our analytical models, as we assume a Gaussian-like symmetric solution of the wave packet

- The odd-order (like third-order) dispersion effect is trivial to the conservation laws.
- In physical space, the dispersive process of the wave groups will not present an asymmetric feature.
- It highlights the need of an asymmetric solution that reflects the influence of the odd-order effects.

Asymmetric features-Airy function solution

The linearized NLS equation with dispersion up-to third-order

$$\frac{\partial U}{\partial T} + \left(\mathrm{i}\frac{\partial^2}{\partial X^2} - \mathrm{i}\frac{\partial^2}{\partial Y^2} - \sqrt{2}\frac{\partial^3}{\partial X^3} + 3\sqrt{2}\frac{\partial^3}{\partial X \partial Y^2}\right) U = 0.$$

Consider a two-dimensional initial Gaussian spectrum

$$U(X,Y,T) = \frac{1}{\sqrt{2\pi}}\frac{A}{S_X S_Y}\int_{-\infty}^{\infty} \exp\left(\mathrm{i}K_Y Y - \mathrm{i}K_Y^2 T\right)\,\mathrm{d}K_Y \times \int_{-\infty}^{\infty} \exp\left(-\frac{K_X^2}{2S_X^2} - \frac{K_Y^2}{2S_Y^2}\right) \exp\left\{\mathrm{i}\left[K_X X + \left(K_X^2 - \sqrt{2}\,K_X^3\right)T + 3\sqrt{2}K_X K_Y^2 T\right]\right\}\,\mathrm{d}K_X$$

Introduce the following variable

$$\hat{K}_X = K_X - \frac{1}{3\sqrt{2}} - \frac{\mathrm{i}}{6\sqrt{2}\,S_X^2\,T} \qquad \alpha = \left(\frac{1}{3\sqrt{2}T}\right)^{1/3}, \quad \tilde{K}_X = -\frac{\hat{K}_X}{\alpha} \qquad Z = X + \frac{T}{3\sqrt{2}} - \frac{1}{12\sqrt{2}\,S_X^4\,T} + \frac{\mathrm{i}}{3\sqrt{2}\,S_X^2}$$

The solution can be written as

$$U(X,Y,T) = \frac{1}{\sqrt{2\pi}}\frac{A}{S_X S_Y}\exp\left[\frac{(1-2\mathrm{i}S_X^2 T)^3}{216 S_X^6 T^2} - \frac{1-2\mathrm{i}S_X^2 T}{6\sqrt{2}S_X^2 T}X\right] \times \int_{-\infty}^{\infty}\exp\left(\mathrm{i}K_Y Y - \frac{K_Y^2}{2S_X^2} - \frac{K_Y^2}{2S_Y^2}\right) I_U(X,T)\,\mathrm{d}K_Y$$

$$I_U(X,T) = \alpha\int_{-\infty-\frac{\mathrm{i}}{6\sqrt{2}\alpha S_X^2 T}}^{\infty-\frac{\mathrm{i}}{6\sqrt{2}\alpha S_X^2 T}} \exp\left\{\mathrm{i}\left[\tilde{K}_X\left(Z + 3\sqrt{2}K_Y^2 T\right) + \frac{\tilde{K}_X^3}{3}\right]\right\}\,\mathrm{d}\tilde{K}_X = 2\pi\alpha\,\mathrm{Ai}\left[-\alpha\left(Z + 3\sqrt{2}K_Y^2 T\right)\right].$$

Airy function

Analytical solution for directional waves

$$|U(X,Y,T)| = \frac{\sqrt{2\pi}A\Gamma(X,T)}{S_X S_Y\left(3\sqrt{2}T\right)^{1/3}}\left|\int_{-\infty}^{\infty}\exp\left(iK_Y Y - \frac{K_Y^2}{2S_X^2} + \frac{K_Y^2}{2S_Y^2}\right)\mathrm{Ai}\left[-\frac{Z + 3\sqrt{2}K_Y^2 T}{\left(3\sqrt{2}T\right)^{1/3}}\right]\mathrm{d}K_Y\right|$$

Analytical solution for unidirectional waves

$$|U(X,T)| = \frac{\sqrt{2\pi}\,A\,\Gamma(X,T)}{S_X\left(3\sqrt{2}T\right)^{1/3}}\left|\mathrm{Ai}\left\{-\left(\frac{1}{3\sqrt{2}T}\right)^{1/3}\left(X + \frac{T}{3\sqrt{2}} - \frac{1}{12\sqrt{2}S_X^4 T} + \frac{\mathrm{i}}{3\sqrt{2}S_X^2}\right)\right\}\right|$$

where

$$\Gamma(X,T) = \exp\left(-\frac{1}{18S_X^2} + \frac{1}{216S_X^6 T^2} - \frac{X}{6\sqrt{2}S_X^2 T}\right)$$

Here, the constants A, SX and SY will be replaced by A(T), SX(T) and SY(T) predicted by the present analytical models to parameterize the effects of nonlinearity, return flow and fourth-order dispersion.

Asymmetric features-unidirectional and directional waves

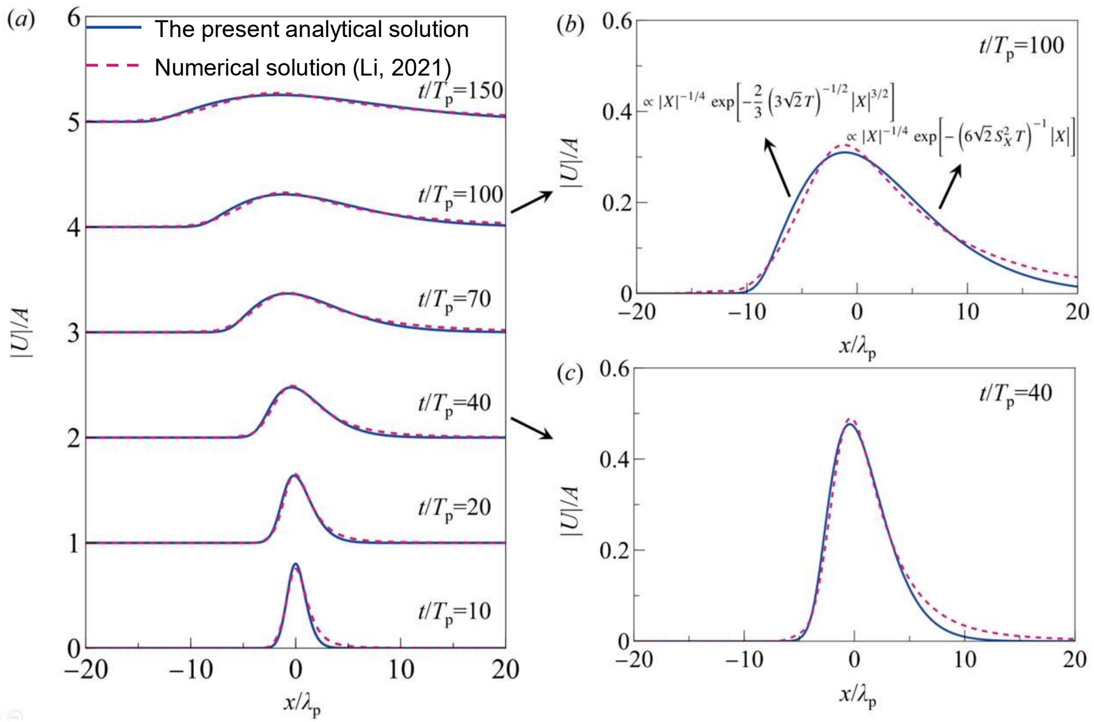


Fig 6. comparison of the wave packet predicted by the analytical solution (blue solid line) and numerical simulations based on (Li, 2021) (red dashed line) for A_f/S_{xf} =0.77 and S_{xf} =0.10 at different time instants between t/T_p =10 and 150 (panel a). Detailed views at t/T_p =100 and 40 are in panels b and c.

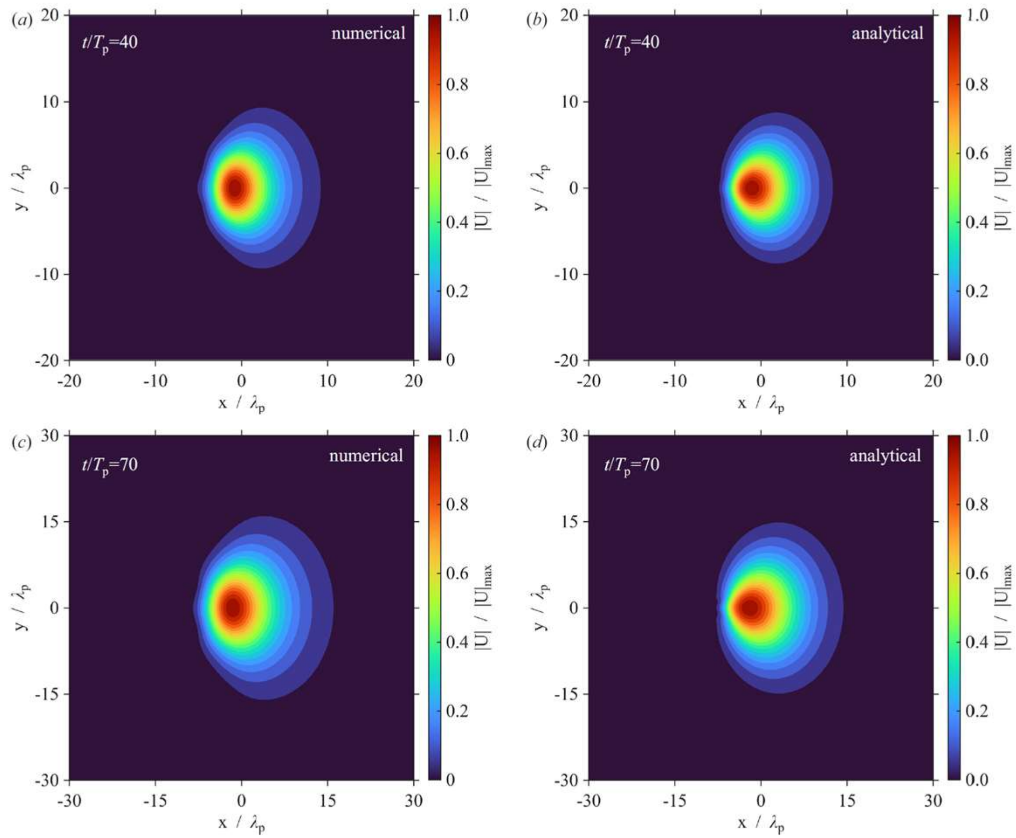


Fig 7. Contour plots of the normalised wave packet from numerical simulations based on Li (2021) (a,c) and the analytical solution (b,d) at t/T_p= 40 and 70 for the case S_{xf}/S_{yf} = 1.94 , A_f/S_{xf} =0.6, and S_{xf} =0.10.

Asymmetric features-analytical quantification

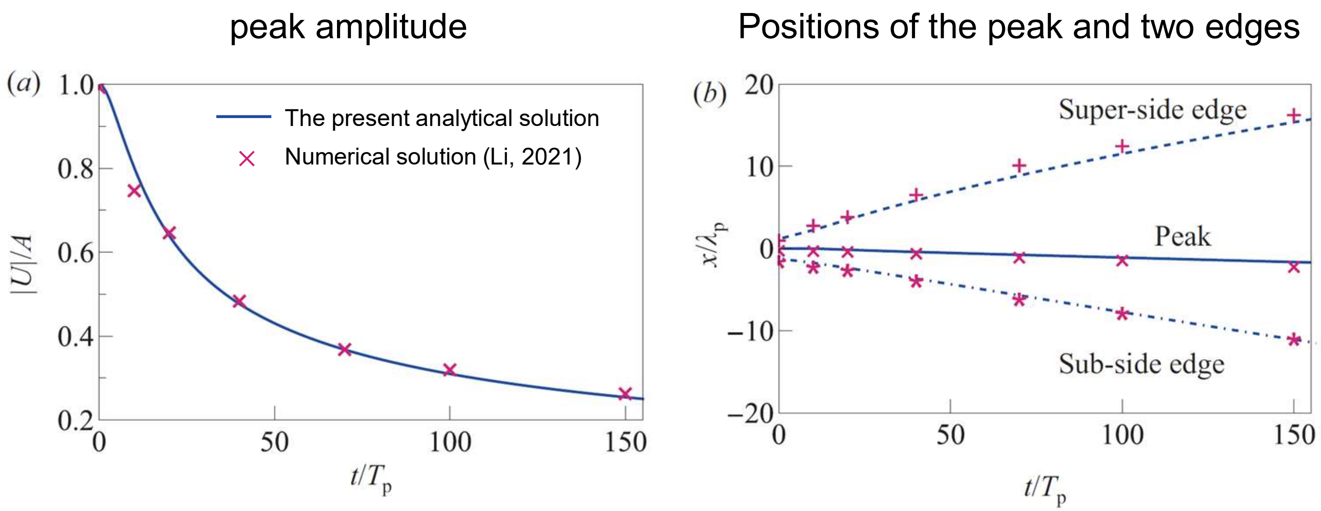


Fig 8. Comparison of the temporal evolution of the wave packet between the analytical solution (blue lines) and numerical simulations based on Li (2021) (red symbols) for the case A_f/S_{xf} =0.77 and S_{xf} =0.10. (a) Peak amplitude; (b) positions of the peak and of the edges on the super- and sub-sides.

Conclusions

This work serves as an extension of Adcock and Taylor (2009) and Adcock et al. (2012).

- Provide a parameterization of key parameters, i.e. amplitude, bandwidth, spreading angle, incorporating higher-order dispersion and return flow.

- Emphasize the influence of third-order dispersion on the modification to the shape of the wave packet, which combined with the analytical models, can give a good prediction on the dispersive waves in broader-banded scenarios.

- Propose several analytical quantifications on the asymmetric features of wave groups, e.g. peak shift, decay rates.

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Thank you for your time!

Questions are welcome.

Splitting of wave packets in broad-banded sea states

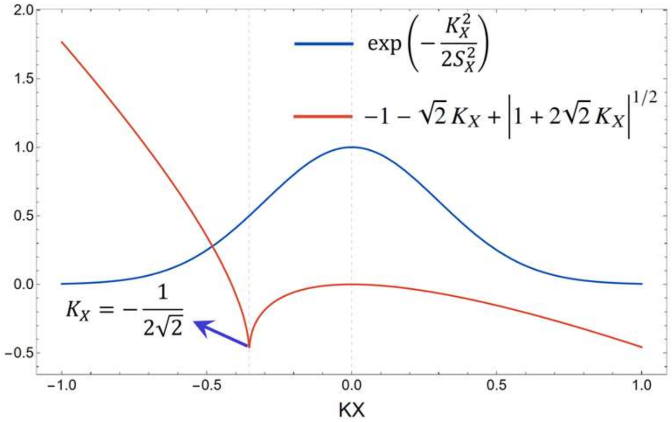
We consider an initial Gaussian distribution with a broad bandwidth, the solution considering an exact linear dispersion is

$$U(X,T) = \frac{1}{\sqrt{2\pi}} \frac{A}{S_X} I_v$$

$$I_v = \int_{-\infty}^{\infty} \exp\left(\mathrm{i} K_X X - \frac{K_X^2}{2S_X^2}\right) \exp\left\{-\mathrm{i}\left[-1-\sqrt{2}\,K_X + \left|1+2\sqrt{2}\,K_X\right|^{1/2}\right] T\right\} \mathrm{d}K_X$$

$$I_v = \underbrace{I_{v1}}_{\text{split-off wave}} + \underbrace{I_{v2}}_{\text{main wave}}$$

$$\int_{-\infty}^{-\frac{1}{2\sqrt{2}}} \cdots \qquad \int_{\frac{1}{2\sqrt{2}}}^{+\infty} \cdots$$



We can conduct an asymptotic analysis to the integrals, using the stationary phase method (Mei et al., 2005), leading to

$$|U_{v1}(X,T)| = \frac{A\Omega^-}{\sqrt{\pi} S_X T^2} \left| \frac{\mathrm{i}}{\Pi} + \frac{\mathrm{e}^{\mathrm{i}/\Pi}\sqrt{\pi}}{(\mathrm{i}\,\Pi)^{3/2}} \left[\mathrm{i} + \mathrm{erfi}\left(\frac{1}{\sqrt{\mathrm{i}\,\Pi}}\right)\right]\right|$$

$$|U_{v2}(X,T)| = \frac{A\Omega^+}{\sqrt{\pi}\, S_X T^2} \left| \frac{\mathrm{i}}{\Pi} + \frac{(1+\mathrm{i})e^{-\mathrm{i}/\Pi}\sqrt{\pi/2}}{\Pi^{3/2}} \,\mathrm{erfc}\left(\frac{(-1)^{3/4}}{\sqrt{\Pi}}\right)\right|$$

$$\Omega^{\mp} = \exp\left\{-\frac{1}{16S_X^2}\left[1 \mp \left(\frac{\sqrt{2}\,T}{\sqrt{2}\,T + X}\right)^2\right]^2\right\}, \qquad \Pi = \frac{2T+\sqrt{2}X}{T^2}$$

Splitting of wave packets in broad-banded sea states

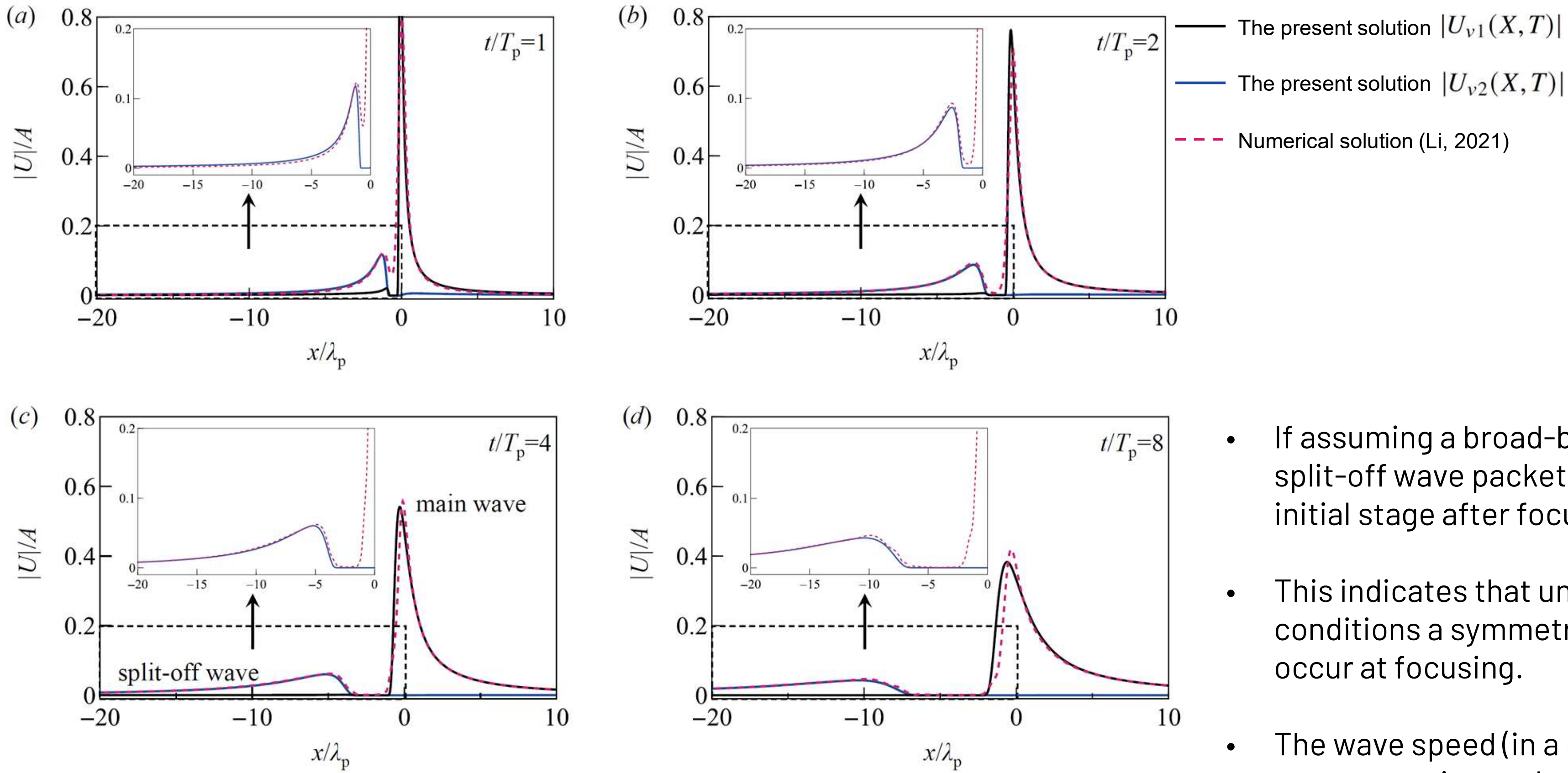


Fig 9. Comparison of the wave packet between numerical simulations based on Li (2021) (red dashed line), the analytical solution for the split-off wave (blue solid line), and the analytical solution for the main wave packet for A_f/S_{xf} =0.77 and S_{xf} =0.30 at different time instants: (a) t/T_p =1, (b) t/T_p =2, (c) t/T_p =4, and (d) t/T_p =8.

- If assuming a broad-banded Gaussian input, a split-off wave packet will be generated at the initial stage after focusing (only once).
- This indicates that under broad-banded conditions a symmetric distribution will not occur at focusing.
- The wave speed (in a non-inertial coordinate system moving at the group velocity) of the split-off wave will be influenced by bandwidth.